THE BEST CONSTANT IN A FRACTIONAL HARDY INEQUALITY

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ABSTRACT. We prove an optimal Hardy inequality for the fractional Laplacian on the half-space.

1. MAIN RESULT AND DISCUSSION

Let $0 < \alpha < 2$ and $d = 1, 2, \ldots$. The purpose of this note is to prove the following Hardy-type inequality in the half-space $D = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$.

Theorem 1. *For every $u \in C_c(D)$,*

$$\frac{1}{2}\int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,\alpha}\int_D u^2(x)\,x_d^{-\alpha}dx\,, \tag{1}$$

where

$$\kappa_{d,\alpha} = \frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{\alpha+d}{2})}\,\frac{B\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right) - 2^\alpha}{\alpha 2^\alpha}\,, \tag{2}$$

and (1) *fails to hold for some $u \in C_c(D)$ if $\kappa_{d,\alpha}$ is replaced by a bigger constant.*

Here B is the Euler beta function, and $C_c(D)$ denotes the class of all the continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ with compact support in D. On the right-hand side of (1) we note the infinite measure $x_d^{-\alpha}dx$, where x_d equals the distance of $x = (x_1, \ldots, x_d) \in D$ to the complement of D. Analogous Hardy inequalities, involving the distance to the complement of rather general domains, and arbitrary positive exponents of integrability of functions u, were proved with *rough* constants in [16] (see also [33, 14, 17]). Thus the focus in Theorem 1 is on optimality of $\kappa_{d,\alpha}$. We note that $\kappa_{d,1} = 0$ and $\kappa_{d,\alpha} > 0$ if $\alpha \neq 1$ (see the proof of Lemma 2).

Theorem 1 may be viewed as an application of ideas of Ancona [1] and Fitzsimmons [19]. Indeed, consider the Dirichlet form $\mathcal{E}$, with domain $Dom(\mathcal{E})$, and the generator $\mathcal{L}$, with domain $Dom(\mathcal{L})$, of a symmetric Markov process ([22], [35], [29]), and a function $w > 0$, and a measure $\nu \geq 0$ on the state space. The following result was proved by Fitzsimmons in [19].

$$\text{If } \mathcal{L}w \leq -w\nu \quad \text{then} \quad \mathcal{E}(u,u) \geq \int u^2\,d\nu\,, \quad u \in Dom(\mathcal{E})\,. \tag{3}$$

Thus, every superharmonic function w (i.e. $w \geq 0$ such that $\mathcal{L}w \leq 0$) yields a Hardy-type inequality with integral weight $\nu = -\mathcal{L}w/w$. For instance, in the proof

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of Theorem 1 we will use $w(x) = x_d^{(\alpha-1)/2}$. Full details of (3), and a converse result are given in [19, Theorem 1.9]. Recall that

$$\mathcal{E}(u,v) = -(Lu,v)\,, \qquad \text{if } u \in Dom(\mathcal{L})\,,\ v \in Dom(\mathcal{E})\,,$$

([22], [35]). Therefore, equality holds in (3) if $u = w \in Dom(\mathcal{L})$, see [19, (1.13.c)], (9). If $w \notin Dom(\mathcal{L})$, or $\mathcal{L}w$ does not belong to the underlying L^2 space, then, as we shall see, the optimality of $\nu = -\mathcal{L}w/w$ critically depends on the choice of w.

According to [7], the Dirichlet form of the censored α-stable process in D is

$$\mathcal{C}(u,v) = \frac{1}{2}\mathcal{A}_{d,-\alpha}\int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dx\,dy\,,$$

with core $C_c^\infty(D)$ (smooth functions in $C_D(D)$), the Lebesgue measure as the reference measure, and the following *regional* fractional Laplacian on D as the generator ([7, (3.12)], [25, 26]):

$$\Delta_D^{\alpha/2}u(x) = \mathcal{A}_{d,-\alpha}\lim_{\varepsilon\to 0^+}\int_{D\cap\{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,dy\,.$$

Here $\mathcal{A}_{d,-\alpha} = \Gamma((d+\alpha)/2)/(2^{-\alpha}\pi^{d/2}|\Gamma(-\alpha/2)|)$, Clearly, (1) is equivalent to

$$\mathcal{C}(u,u) \geq \mathcal{A}_{d,-\alpha}\kappa_{d,\alpha}\int_D u^2(x)\ x_d^{-\alpha}dx\,. \tag{4}$$

Recall that the Dirichlet form of the stable process killed when leaving D is

$$\mathcal{K}(u,v) = \frac{1}{2}\mathcal{A}_{d,-\alpha}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dx\,dy\,,$$

with core $C_c^\infty(D)$, the Lebesgue measure as the reference measure, and the fractional Laplacian (on $\mathbb{R}^d$) as the generator,

$$\Delta^{\alpha/2}u(x) = \mathcal{A}_{d,-\alpha}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^d\cap\{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,dy$$

(see, e.g., [7]). Decomposing $\mathbb{R}^d = D\cup D^c$, one obtains

$$\mathcal{K}(u,u) = \mathcal{C}(u,u) + \int_D u^2(x)\kappa_D(x)dx\,, \quad u\in C_c^\infty(D)\,,$$

where (the density of the killing measure for D is)

$$\kappa_D(x) = \int_{D^c}\mathcal{A}_{d,-\alpha}|x-y|^{-d-\alpha}\,dy = \frac{1}{\alpha}\mathcal{A}_{d,-\alpha}\frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{\alpha+d}{2})}\ x_d^{-\alpha}\,,$$

see [7, (2.3), (5.4)-(5.6)]. It follows from (4) and Theorem 1 that

$$\begin{aligned}
\mathcal{C}(u,u) \quad &\geq \quad \mathcal{A}_{d,-\alpha}\Big(\kappa_{d,\alpha} + \frac{1}{\alpha}\frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{\alpha+d}{2})}\Big)\int_D u^2(x)x_d^{-\alpha}dx \\
&= \quad \frac{\Gamma^2(\frac{1+\alpha}{2})}{\pi}\int_D u^2(x)x_d^{-\alpha}dx\,,
\end{aligned} \tag{5}$$

for all $u\in C_c^\infty(D)$, and the constant $\Gamma^2(\frac{1+\alpha}{2})/\pi$ is the best possible.

We like to note that in some respects, the censored stable process is a better analogue of the killed Brownian motion than the killed stable process is (see [14, 7, 32], and [37, 31]). We suggest the former as a possible setup for studying Dirichlet boundary value problems for non-local integro-differential operators and the corresponding stochastic processes ([30], [38]) on subdomains of $\mathbb{R}^d$ ([3]), beyond the

"convolutional" case of the whole of $\mathbb{R}^d$ ([21, 4]). In this connection, we refer to [25, 26, 24] for Green-type formulas for the censored process.

The reader interested in fractional Hardy inequalities may consult [34, 27, 33, 14, 16, 17]. In particular, (1) improves a part of the (one-dimensional) result given in [33, Theorem 2]. The fractional Hardy inequality on the whole of $\mathbb{R}^d$ is known as Hardy-Rellich inequality, and the best constant in this inequality was calculated in [28, 39] (see also [4] for Pitt's inequality). As seen in [16], the asymptotics of the measure $\mathrm{dist}(x, D^c)^{-\alpha}dx$ agrees well with the homogeneity of the kernel $|y-x|^{-d-\alpha}$ in (1). Noteworthy, if $\alpha \le 1$ and D is a *bounded* Lipschitz domain, then the best constant in (1) is zero ([16]).

We like to make a few further remarks. Theorem 1 and the results obtained to date for Laplacian and fractional Laplacian suggest possible strengthenings to weights with additional terms of lower-order boundary asymptotics ([11, 23, 4]), and extensions to other specific or more general domains ([36, 4]). To discuss the latter problem, we consider open $\Omega \subset D$, and its Hardy constant, $\kappa(\Omega)$, defined as the largest number such that

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy \ge \kappa(\Omega)\int_\Omega \frac{u^2(x)}{\mathrm{dist}(x,\Omega^c)^\alpha}dx\,, \quad u \in C_c(\Omega)\,.$$

Note that $\kappa(\Omega) > 0$ if Ω is a bounded Lipschitz domain and $\alpha > 1$ ([16]). Let $u \in C_c(\Omega) \subset C_c(D)$. We have

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy \le \frac{1}{2}\int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy\,,$$

and

$$\int_\Omega \frac{u^2(x)}{\mathrm{dist}(x,\Omega^c)^\alpha}dx \ge \int_D \frac{u^2(x)}{x_d^\alpha}dx\,,$$

thus $\kappa(\Omega) \le \kappa_{d,\alpha}$. We conjecture that $\kappa(\Omega) = \kappa_{d,\alpha}$ for $\alpha \in (1,2)$ and *convex* Ω, see [36, Theorem 11] for case of the Dirichlet of Laplacian.

Examining (2) we see that $\kappa_{d,\alpha} \to \infty$ if $\alpha \to 2$. This corresponds to the fact that the only function $u \in C_c(\Omega)$ for which the left hand side of (1) is finite for $\alpha = 2$ is the zero function, see [12, 16]. However, $\mathcal{A}_{d,-\alpha}\kappa_{d,\alpha} \to 1/4$ and $\Gamma^2(\frac{1+\alpha}{2})/\pi \to 1/4$ as $\alpha \to 2$, an agreement with the classical Hardy inequality for Laplacian ([11]) related to the fact that for $u \in C_c^\infty(D)$, $\Delta_D^{\alpha/2}u \to \Delta u$ and $\mathcal{C}(u,u) \to -\int \Delta u(x)u(x)dx = \int |\nabla u(x)|^2 dx$ as $\alpha \to 2$ (the latter holds by Taylor's expansion of order 2, and a similar result is valid for $\mathcal{K}$). For the vast literature concerning optimal weights and constants in the classical Hardy inequalities, and their applications we refer to [5, 15, 11, 20, 23, 18, 2].

Our primary motivation to study Hardy inequalities for non-local Dirichlet forms stems from the fact that the converse of (3) stated in [19, Theorem 1.9] allows for a construction of superharmonic functions, or barriers ([1]), when a Hardy inequality is given. These functions may then be used to investigate transience and boundary behavior of the underlying Markov processes ([1], [7], [17], [14]). In particular, we expect that the results of [16, 17] may be used to obtain, for the anisotropic stable ([10, 9]) censored processes, the ruin probabilities generalizing [7, Theorem 5.10], and to develop the boundary potential theory on Lipschitz domains ([7, 26, 24]) in analogy with those of the killed stable processes ([8, 13, 6]). We also like to mention the connection of optimal Hardy inequalities with critical Schrödinger perturbations and the so-called ground state representation [21].

Despite the general context mentioned above, the paper is essentially self-contained and purely analytic. In particular we directly derive Fitzsimmons' ratio measure by a simple manipulation with quadratic expressions, (9), not unrelated to the ground state representation of [21, (4.2)]. Theorem 1 is proved below in this section. In Section 2 we calculate auxiliary integrals.

In what follows, $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$ denotes the Euclidean norm of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, and $B(x, r)$ denotes the Euclidean ball of radius $r > 0$ centered at x. For $d \geq 2$ we occasionally write $x = (x', x_d)$, where $x' = (x_1, \ldots, x_{d-1})$, and we let $\|x'\| = \max_{k=1,\ldots,d-1} |x_k|$, the supremum norm on $\mathbb{R}^{d-1}$.

Proof of Theorem 1. For $u, v \in C_c^\infty(D)$ we define (Dirichlet form)

$$\mathcal{E}(u,v) = \frac{1}{2}\int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dx\,dy\,,$$

and (its generator)

$$\mathcal{L}u(x) = \lim_{\varepsilon\to 0^+}\int_{D\cap\{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\,dy\,,$$

so that $\mathcal{A}_{d,-\alpha}\mathcal{E} = \mathcal{C}$, $\mathcal{A}_{d,-\alpha}\mathcal{L} = \Delta_D^{\alpha/2}$, and $\mathcal{E}(u,u)$ equals the left-hand side of (1).

Let $p \in (-1, \alpha)$, $x = (x_1, \ldots, x_d) \in D$,

$$w_p(x) = x_d^p\,.$$

By [7, (5.4) and (5.5)],

$$\mathcal{L}w_p(x) = \gamma(\alpha,p)\,\frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{\alpha+d}{2})}\,x_d^{-\alpha}\,w_p(x)\,, \tag{6}$$

where the (absolutely convergent) integral

$$\gamma(\alpha,p) = \int_0^1 \frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{1+\alpha}}\,dt\,, \tag{7}$$

is negative if $p(\alpha-p-1) > 0$. Guided by the discussion in Section 1 we let

$$\nu(x) = \frac{-\mathcal{L}w_p(x)}{w_p(x)} = -\gamma(\alpha,p)\frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{\alpha+d}{2})}\;x_d^{-\alpha}\,. \tag{8}$$

Since, for each $t \in (0,1)$, the function

$$p \mapsto \frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{1+\alpha}}$$

is convex and symmetric with respect to $(\alpha-1)/2$, therefore $p \mapsto \gamma(\alpha,p)$ has a non-positive minimum at $p = (\alpha-1)/2$. By Lemma 2 below, (8), and (3), we obtain (1) for $u \in C_c^\infty(D) \subset Dom(\mathcal{C})$, with $\kappa_{d,\alpha}$ given by (2). The case of general $u \in C_c(D)$ is obtained by an approximation.

Since the setups of [19] and [7] are rather complex, we like to give the following elementary proof of (1). Let $w = w_{(\alpha-1)/2}$, $u \in C_c(D)$, $x, y \in D$. We have

$$\begin{aligned}(u(x)-u(y))^2 \quad &+ \quad u^2(x)\frac{w(y)-w(x)}{w(x)} + u^2(y)\frac{w(x)-w(y)}{w(y)}\\ &= \quad w(x)w(y)\left[u(x)/w(x)-u(y)/w(y)\right]^2 \geq 0\,.\end{aligned} \tag{9}$$

We integrate (9) against the symmetric measure $1_{|y-x|>\varepsilon}|x-y|^{-d-\alpha}\,dx\,dy$, and we let $\varepsilon \to 0^+$. According to the calculations above,

$$\begin{aligned}\frac{1}{2}\int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy &\geq \int_D u^2(x)\lim_{\varepsilon\to 0^+}\int_{\{y\in D:\,|y-x|>\varepsilon\}} \frac{w(x)-w(y)}{|y-x|^{d+\alpha}}\,dy\,\frac{dx}{w(x)}\\ &= \kappa_{d,\alpha}\int_D u^2(x)\,x_d^{-\alpha}\,dx\,.\end{aligned}$$

To complete the proof we will verify the optimality of $\kappa_{d,\alpha}$. In what follows we denote $\mathbf{p} = \frac{\alpha-1}{2}$. If $\alpha \geq 1$ then we consider functions v_n such that

(i) $v_n = 1$ on $[-n^2, n^2]^{d-1} \times [\frac{1}{n}, 1]$,
(ii) $\operatorname{supp} v_n \subset [-n^2-1, n^2+1]^{d-1} \times [\frac{1}{2n}, 2]$,
(iii) $0 \leq v_n \leq 1$, $|\nabla v_n(x)| \leq c x_d^{-1}$ and $|\nabla^2 v_n(x)| \leq c x_d^{-2}$ for $x \in D$.

If $\alpha < 1$ then we stipulate

(i') $v_n = 1$ on $[-n^2, n^2]^{d-1} \times [1, n]$,
(ii') $\operatorname{supp} v_n \subset [-n^2-n, n^2+n]^{d-1} \times [\frac{1}{2}, 2n]$,
(iii) $0 \leq v_n \leq 1$, $|\nabla v_n(x)| \leq c x_d^{-1}$ and $|\nabla^2 v_n(x)| \leq c x_d^{-2}$ for $x \in D$.

We define (for any $\alpha \in (0,2)$),

$$u_n(x) = v_n(x) x_d^{\mathbf{p}}\,. \tag{10}$$

We have

$$\int_D \frac{u_n(x)^2}{x_d^\alpha}\,dx \geq \int_{\{x:\,\|x'\|\leq n^2,\ \frac{1}{n}<x_d<1\}} \frac{x_d^{2\mathbf{p}}}{x_d^\alpha}\,dx = (2n^2)^{d-1}\log n.$$

Thus, by Lemma 4 below, $\kappa_{d,\alpha}$ may not be replaced in (1) by a bigger constant. □

2. Appendix

Lemma 2. *For $0 < \alpha < 2$,*

$$\gamma(\alpha, \frac{\alpha-1}{2}) = -\frac{1}{\alpha}\left[B(\frac{1+\alpha}{2}, \frac{2-\alpha}{2})2^{-\alpha} - 1\right]. \tag{11}$$

Proof. Since

$$\gamma(\alpha,p) = \int_0^1 \frac{t^p - t^{\alpha-1} - 1 + t^{\alpha-p-1}}{(1-t)^{1+\alpha}}\,dt\,,$$

we are led to considering

$$B_\kappa(a,b) = \int_0^\kappa t^{a-1}(1-t)^{b-1}\,dt\,.$$

Here and below $a > 0$, $b > -2$, and $0 \leq \kappa < 1$. We will also assume that $b \neq 0, 1$.

Using $t^{a-1} = t^{a-1}(1-t) + t^a$, and integration by parts, we get

$$B_\kappa(a,b) = \frac{a+b}{b}B_\kappa(a,b+1) - \frac{1}{b}\kappa^a(1-\kappa)^b\,,$$

hence

$$B_\kappa(a,b) = \frac{a+b}{b}\left(\frac{a+b+1}{b+1}B_\kappa(a,b+2) - \frac{1}{b+1}\kappa^a(1-\kappa)^{b+1}\right) - \frac{1}{b}\kappa^a(1-\kappa)^b\,.$$

Clearly, $\gamma(\alpha,p) = \lim_{\kappa\to 1^-}\left[B_\kappa(p+1,-\alpha) - B_\kappa(\alpha,-\alpha) - B_\kappa(1,-\alpha) + B_\kappa(\alpha-p,-\alpha)\right]$. For $\alpha \neq 1$ we have,

$$\begin{aligned}
&B_\kappa(p+1,-\alpha) - B_\kappa(\alpha,-\alpha) - B_\kappa(1,-\alpha) + B_\kappa(\alpha-p,-\alpha) = \frac{1}{\alpha(\alpha-1)}\times\\
&\{(p+1-\alpha)(p+1-\alpha+1)B_\kappa(p+1,2-\alpha) - (\alpha-\alpha)(\alpha-\alpha+1)B_\kappa(\alpha,2-\alpha)\\
&-(1-\alpha)(1-\alpha+1)B_\kappa(1,2-\alpha) + (\alpha-p-\alpha)(\alpha-p-\alpha+1)B_\kappa(\alpha-p,2-\alpha)\}\\
&+\frac{(1-\kappa)^{1-\alpha}}{\alpha(\alpha-1)}\left[-(p+1-\alpha)\kappa^{p+1} + (\alpha-\alpha)\kappa^\alpha + (1-\alpha)\kappa^1 - (\alpha-p-\alpha)\kappa^{\alpha-p}\right]\\
&+\frac{(1-\kappa)^{-\alpha}}{-\alpha}\left[-\kappa^{p+1}+\kappa^\alpha+\kappa^1-\kappa^{\alpha-p}\right].
\end{aligned}$$

All expressions in the square brackets, and their derivative, vanish at $\kappa = 1$. Thus, they do not contribute to the limit as $\kappa \to 1$. For $\alpha \neq 1$ we get

$$\begin{aligned}
\gamma(\alpha,p) &= \frac{1}{\alpha(\alpha-1)}\{(p+1-\alpha)(p+2-\alpha)B(p+1,2-\alpha)\\
&\quad -(1-\alpha)(2-\alpha)B(1,2-\alpha) + p(p-1)B(\alpha-p,2-\alpha)\}\,. \qquad (12)
\end{aligned}$$

By the duplication formula $\Gamma(2z) = (2\pi)^{-1/2}\,2^{2z-1/2}\,\Gamma(z)\,\Gamma(z+1/2)$ with $2z = 2-\alpha$, for $p = (\alpha-1)/2$, this equals

$$\begin{aligned}
&\frac{1}{\alpha}\left[-\frac{3-\alpha}{2}B(\frac{\alpha+1}{2},2-\alpha)+1\right] = \frac{1}{\alpha}\left[-\Gamma(\frac{\alpha+1}{2})\Gamma(2-\alpha)/\Gamma(\frac{3-\alpha}{2})+1\right]\\
&= \frac{1}{\alpha}\left[-\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{2-\alpha}{2})/\Gamma(\frac{1}{2})2^{1-\alpha}+1\right] = -\frac{1}{\alpha}\left[B(\frac{\alpha+1}{2},\frac{2-\alpha}{2})2^{-\alpha}-1\right].
\end{aligned}$$

We thus proved (11) for $\alpha \neq 1$. The case of $\alpha = 1$ is trivial. In fact, $\gamma(1,0) = 0$. $\square$

Lemma 3. *Let $-1 < r < \alpha < 2$ and $\alpha > 0$. There exists a constant c such that*

$$\int_{D\setminus B(x,a)} \frac{y_d^r}{|x-y|^{d+\alpha}}\,dy \le c a^{-\alpha}(a \vee x_d)^r$$

for every $a > 0$ and $x \in D$.

Proof. Let $B(x,s,t) = B(x,t)\setminus B(x,s)$. If $a \ge x_d/2$ then

$$\begin{aligned}
\int_{D\setminus B(x,a)} \frac{y_d^r}{|x-y|^{d+\alpha}}\,dy &\le c\sum_{k=0}^\infty \int_{D\cap B(x,2^k a,2^{k+1}a)} \frac{y_d^r}{(2^k a)^{d+\alpha}}\,dy\\
&\le c'\sum_{k=0}^\infty (2^k a)^{r-\alpha} = c'' a^{r-\alpha}.
\end{aligned}$$

If $a < x_d/2$ then

$$\int_{D\cap B(x,a,x_d)} \frac{y_d^r}{|x-y|^{d+\alpha}}\,dy \le c x_d^r\, a^{-\alpha}\,,$$

and, by first part of the proof,

$$\int_{D\setminus B(x,x_d)} \frac{y_d^r}{|x-y|^{d+\alpha}}\,dy \le c x_d^{r-\alpha}.$$

This ends the proof. $\square$

Recall that $\mathbf{p} = \frac{\alpha-1}{2}$, and u_n is defined by (10).

Lemma 4. *There exists a constant c independent of n, such that*
$$\int_D\int_D \frac{(u_n(x)-u_n(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx \le cn^{2(d-1)} + 2\kappa_{d,\alpha}\int\limits_D u_n^2(x)\,x_d^{-\alpha}\,dx.$$

Proof. To simplify the notation we let $K_n = \operatorname{supp} u_n$ and $u = u_n$, $v = v_n$. By (9) and (6) we have
$$\begin{aligned}\int\limits_D\int\limits_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy &= 2\kappa_{d,\alpha}\int\limits_D u^2(x)\,x_d^{-\alpha}\,dx \\ &\quad + \int\limits_D\int\limits_D \frac{(v(x)-v(y))^2}{|x-y|^{d+\alpha}}\,w(x)w(y)\,dx\,dy.\end{aligned}$$

We will estimate the latter (double) integral by $cn^{2(d-1)}$, by splitting it into the sum of the following six integrals $I_1+\ldots+I_6$.

We will first consider the case of $\alpha \ge 1$.

If $x \in K_n$ and $y \in B(x, \frac{1}{4n})$, then $|v(x)-v(y)| \le c|x-y|x_d^{-1}$, as follows from (ii) and (iii). We thus have
$$\begin{aligned} I_1 &= \int_D\int_{B(x,\frac{1}{4n})} \frac{(v(x)-v(y))^2}{|x-y|^{d+\alpha}}\,w(x)w(y)\,dy\,dx \\ &\le 2\int_{K_n}\int_{B(x,\frac{1}{4n})} \frac{(v(x)-v(y))^2}{|x-y|^{d+\alpha}}\,w(x)w(y)\,dy\,dx \\ &\le c\int_{K_n}\int_{B(x,\frac{1}{4n})} \frac{x_d^{2\mathbf{p}-2}}{|x-y|^{d+\alpha-2}}\,dy\,dx \\ &\le c'n^{2(d-1)}. \end{aligned}$$

A similar argument gives
$$I_2 = \int_{\{x:\,x_d\ge\frac12\}}\int_{B(x,\frac14)} \frac{(v(x)-v(y))^2}{|x-y|^{d+\alpha}}\,w(x)w(y)\,dy\,dx \le cn^{2(d-1)}.$$

We then have by Lemma 3 for $a = 1/4$ and $r = \mathbf{p}$
$$\begin{aligned} I_3 &= \int_D\int_{D\setminus B(x,\frac14)} \frac{(v(x)-v(y))^2}{|x-y|^{d+\alpha}}\,dy\,dx \\ &\le \int_{K_n}\int_{D\setminus B(x,\frac14)} \frac{c}{|x-y|^{d+\alpha}}\,w(x)w(y)\,dy\,dx \\ &\le c'n^{2(d-1)}. \end{aligned}$$

If $d \ge 2$ then we consider $P_n = \{x \in \mathbb{R}^d : \|x'\| \ge n^2-1,\ 0 < x_d < \frac12\}$ and $P_n^0 = P_n \cap \{x \in \mathbb{R}^d : \|x'\| < n^2+\frac54\}$. We obtain
$$\begin{aligned} I_4 &= \int_{P_n}\int_{D\cap B(x,\frac{1}{4n},\frac14)} \frac{(v(x)-v(y))^2}{|x-y|^{d+\alpha}}\,w(x)w(y)\,dy\,dx \\ &\le \int_{P_n^0}\int_{D\setminus B(x,\frac{1}{4n})} \frac{c}{|x-y|^{d+\alpha}}\,dy\,dx \\ &\le c'|P_n^0|n^\alpha \le c''n^{2(d-1)}. \end{aligned}$$

We let $R_n = \{x \in \mathbb{R}^d : \|x'\| < n^2 - 1,\ 0 < x_d < \frac{2}{n}\}$ if $d \geq 2$, and we let $R_n = \{x \in \mathbb{R} : 0 < x < \frac{2}{n}\}$ if $d = 1$. We have

$$\begin{aligned} I_5 &= \int_{R_n} \int_{D \cap B(x, \frac{1}{4n}, \frac{1}{4})} \frac{(v(x) - v(y))^2}{|x-y|^{d+\alpha}} w(x)w(y)\, dy\, dx \\ &\leq \int_{R_n} \int_{D \setminus B(x, \frac{1}{4n})} \frac{c y_d^{\mathbf{p}} (\frac{1}{n})^{\mathbf{p}}}{|x-y|^{d+\alpha}}\, dy\, dx \leq c' n^{2(d-1)}. \end{aligned}$$

In the last inequality above we have used Lemma 3 with $a = \frac{1}{4n}$ and $r = \mathbf{p}$.

We define $L_n = \{x \in \mathbb{R} : \frac{2}{n} \leq x < \frac{1}{2}\}$ in dimension $d = 1$, and for $d \geq 2$ we let $L_n = \{x \in \mathbb{R}^d : \|x'\| < n^2 - 1,\ \frac{2}{n} \leq x_d < \frac{1}{2}\}$. We have

$$\begin{aligned} I_6 &= \int_{L_n} \int_{D \cap B(x, \frac{1}{4n}, \frac{1}{4})} \frac{(v(x) - v(y))^2}{|x-y|^{d+\alpha}} w(x)w(y)\, dy\, dx \\ &\leq \int_{L_n} \int_{\{y : 0 < y_d < \frac{1}{n}\}} \frac{w(x)w(y)}{|x-y|^{d+\alpha}}\, dy\, dx \end{aligned}$$

For $d \geq 2$ and $x \in L_n$ we have

$$\begin{aligned} &\int\limits_{\{y:0<y_d<\frac{1}{n}\}} \frac{dy}{|x-y|^{d+\alpha}} \leq c \int\limits_{\{y:0<y_d<\frac{1}{n}\}} \frac{dy}{(|x'-y'|^2 + x_d^2)^{(d+\alpha)/2}} \\ &= \frac{c}{n} \left(\int_{\{y' \in \mathbb{R}^{d-1} : |x'-y'| < x_d\}} + \int_{\{y' \in \mathbb{R}^{d-1} : |x'-y'| \geq x_d\}} \right) \frac{dy'}{(|x'-y'|^2 + x_d^2)^{(d+\alpha)/2}} \\ &\leq c' \frac{x_d^{-\alpha-1}}{n}, \end{aligned}$$

thus

$$I_6 \leq c \int_{L_n} \left(\frac{x_d}{n}\right)^{\mathbf{p}} \frac{x_d^{-\alpha-1}}{n}\, dx \leq c' n^{2(d-1)}.$$

The case of $d = 1$ is left to the reader.

We now consider the case of $\alpha < 1$. We have

$$\begin{aligned} I &= \int\limits_D \int\limits_D \frac{(v(x) - v(y))^2}{|x-y|^{d+\alpha}} w(x)w(y)\, dx\, dy \\ &\leq \int_D \int_{B(x, \frac{1}{4})} + \int_{\{x : x_d \geq \frac{n}{2}\}} \int_{B(x, \frac{n}{4})} + \int_D \int_{D \setminus B(x, \frac{n}{4})} + \int_{P_n} \int_{D \cap B(x, \frac{1}{4}, \frac{n}{4})} \\ &\quad + \int_{\{x : 0 < x_d < 2\}} \int_{D \cap B(x, \frac{1}{4}, \frac{n}{4})} + \int_{L_n} \int_{D \cap B(x, \frac{1}{4}, \frac{n}{4})} \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned}$$

where

$$\begin{aligned} P_n &= \{x \in \mathbb{R}^d : \|x'\| \geq n^2 - n,\ 0 < x_d < \frac{n}{2}\}; \\ L_n &= \{x \in \mathbb{R}^d : \|x'\| < n^2 - n,\ 2 \leq x_d < \frac{n}{2}\}, \end{aligned}$$

for $d \geq 2$, and $P_n = \emptyset$, $L_n = (2, \frac{n}{2})$ for $d = 1$. We estimate the integrals I_k in a similar way as for $\alpha \geq 1$. The details are left to the reader. □

Similar but simpler estimates were given in [16] to prove that the Hardy constant of a bounded Lipschitz domain (e.g. of an interval) is zero if $\alpha \leq 1$. We also like to mention that there is an alternative proof of Lemma 4 (not given here), which explicitly uses the fact that that $w^2(x) = x_d^{\alpha-1}$ is harmonic ([7]) for $\Delta_D^{\alpha/2}$. Similarly, the best constant, $1/4$, in the classical Hardy inequality for the half-space D is obtained by considering $w(x) = \sqrt{x_d}$ in Fitzsimmons' ratio $\nu = -\Delta w/w$.

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